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By

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Topic

On the fixed point principles of Banach and Kannan's

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Didications

To my parents, my mother Hayat and my father Belkacem.

To my brothers and sisters, Kamel, Nawri, Rafik, Sami, Adel, and Naaima.

To my husband Tarek.

To my freinds.

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Introduction

Fixed point is one of the most powerful and fruitful tools of modern mathematics and my be a core subject of non linear analysis. In 1886 Poincare was the first to work in this field. Then Brouwerin 1912 proved fixed point theorem for the solution of the equation $f(x) = x$. He also proved fixed point theorem for a square, a sphere and their n-dimensional counter parts which was further extended by Kakutani. Mean while Banach principle came in to existence which was considered as one of the fundamental principle in the field of functional analysis. In 1922 Banach proved that a contraction mapping in the field of a complete metric space possesses a unique fixed point. Later on it was developed by Kannan. Fixed point theory is a interdisciplinary topic which can be applied in various disciplines of mathematics.

The goal of our work is to show the different theorems of fixed point of Banach and Kannan's in different space(metric and generalized metric space, functionnal space). This theorems are very important in mathematics to find a solution of some equation operator.

This memory is diveded into three chapter:

First chapter: In this chapter we speak about some definitions and some basis in metric space and functionnal space who needs in the other chapter.

Second chapter: In this chapter we have presented the theorems of Banach and Kannan in the metric space and functionnal space.

Third chapter: we presented the application of the theorems of Banach and Kannan's.

Chapter 1

Basics and preliminaries

1.1 Introduction

In this chapter we expose fundamental notions and theorems are presented, which will be used in the following chapters, these theorems that are used briefly or intuitively, of raison to guide towards the axis of our work.

1.2 Some fundamental definitions and theorems

1.2.1 Banach space

A normed vector space $(E, \|.\|)$ is called a Banach space, if any so-called Cauchy sequence in E is convergent to a limit that belongs to E; where a sequence$\{x^{(n)}\}_{n=1}^{\infty}$is called

a Cauchy sequence if the following holds:

$$\text{For any } \varepsilon > 0 \text{ there exists } n_0(\varepsilon) \text{ such that } \left\|x^{(m)} - x^{(n)}\right\| < \varepsilon$$

if $m, n \geq n_0(\varepsilon)$.

1.2.2 Metric space

suppose X is a set and d is the real function defined in the cartesian product $X \times X$. then d is called a metric on X , if, and only if, for each $a, b, c \in X$:

· $d(a, b) \geq 0$ for all $a, b \in X$ and $d(a, b) = 0$ if and only if $a = b$,

· $d(a, b) = d(b, a)$,

· $d(a, b) \leq d(a, c) + d(c, b)$.

1.2.3 Completeness

Suppose (X, d) is a metric space and S is a subset of X. We say that S is a complete subset of X, if, and only if, the metric subspace (S, d) of (X, d) is a complete metric space.

1.2.4 Complete metric space:

If (M, d) is a metric space, then there exists a complete metric space (M^*, d^*) and a mapping $h : M \to M^*$such that

1) h is an isometry $d^*(h(x), h(y)) = d(x, y) \quad x, y \in M$

2) $h(M)$ is dense in M^*.

1.2.5 Generalized metric space

Let X be a nonempty set. Suppose that the mapping $d : X \to X$, satisfies :

$(i) d(x, y) \geq 0$,for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

$(ii) d(x, y) = d(y, x)$ for all $x, y \in X$;

$(iii) d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X \backslash \{x, y\}$ [rectangular property].

Then d is called a generalized metric and (X, d) is a generalized metric space.

1.2.6 Sequentially convergent

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence.

1.2.7 Subsequentially convergent

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$

is convergence then $\{y_n\}$ has a convergent subsequence.

1.2.8 Cauchy sequence

A sequence of vectors (x_n) in a normed space is called a Cauchy sequence if for every $\varepsilon > 0$ there exists a number M such that $\|x_n - x_m\| < \varepsilon$

for all $m, n > M$.

1.2.9 Normed vector space

Let X be a vector space over K, ($K = R$ or C), the field of real or complex numbers. A mapping$\|.\| : X \longrightarrow R$ is called a norm, provided that the following conditions holds:

(i) $\|x\| = 0 \Longrightarrow x = 0$,

(ii) $\|\lambda x\| = |\lambda| \, \|x\|, \forall \lambda \in K, \forall x \in X$,

(iii) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$.

If X is a vector space and$\|.\|$ is a norm on X, then the pair $(X, \|.\|)$ is called a normed vector space.

1.2.10 Norme space

A vector space with a norm is called a normed space.

1.2.11 Normal structure

A bounded convex set K in a Banach space X is said to have normal structure if for each convex subset S of K which contains more than one point, there exists $x \in S$ such that

$$sup_{y \in S} \|x - y\| < \delta(S),$$

$\delta(S)$,being the diameter of S.

1.2.12 Definition

A mapping T of a bounded subset K of a Banach space X into itself is said to have property B on K if for every closed convex subset F of K, mapped into itself by T and containing more than one element, there exists $x \in F$ such that

$$\|x - Tx\| < \sup_{y \in F} \|y - Ty\| .$$

Remark 1.2.1

If K has normal structure then a mapping T, having property A on K, of K into itself must have property B on K but not conversely.

1.2.13 Reflexivity

A normed space X is said to be reflexive if the natural embeding mapping $\varphi : X \to X^{**}$ is onto. In this case, we write $X \cong X^{**}$ or $X = X^{**}$

1.2.14 Lipschitzian operator

Let (X, d) be a metric space. A map $F : X \to X$ is said to be Lipschitzian if there exists a constant $\alpha \geq 0$ with

$d(F(x), F(y)) \leq \alpha d(x, y)$ forall $x, y \in X$.

1.2.15 Dense

Suppose X is a metric space and S is a subset of X then S is said to be dense in X, if, and only if

$\overline{S} = X$.

Chapter 2

Comparison between the principle of Banach and Kannan's

2.1 Introduction

In this chapter we discuss about the Banach and Kannan's principles, and we show the difference between them.

2.2 In the case of a functional space

Theorem 2.2.1 ***(Kannan)***

Let T be a mapping of a nonempty bounded closed and convex subset K of a reflexive Banach space X into itself and let T have property A over K. Then if

$$sup_{y \in F} \|y - Ty\| < \delta(F)$$

for every nonempty bounded closed convex subset F of K, containing more than one element and mapped into itself by T, has a unique fixed point in K.

Proof.

Let τ be the family of all closed convex bounded subsets of K, mapped into itself by T. Obviously τ is nonempty. Applying Zorn's lemma, we get a minimal element S in τ, being minimal with respect to being nonempty, bounded closed and convex and invariant under T. If S contains only one element , then that element is a fixed point of T. If not, let S contain more than one element. Now for $x, y \in S$,

$$\|Tx - Ty\| \leq \frac{\|x - Tx\|}{2} + \frac{\|y - Ty\|}{2} \leq \sup_{y \in S} \|y - Ty\| .$$

$T(S)$ is contained in the closed sphere C with Tx as a centre and $sup_{y \in S} \|y - Ty\|$ as radius. Also $S \cap C$ is invariant under T. Therefore by the minimality of S it follows that $S \subset C$ i.e.;

$$\|Tx - Ty\| \leq \sup_{y \in S} \|y - Ty\|$$

for every $y \in S$.

Hence, for any arbitrary but fixed $x \in S$,we have

$$\sup \|Tx - y\| \leq \sup \|y - Ty\| . \tag{2.2.1}$$

Let $S^{'} = \{z \in S : \sup_{y \in S} \|z - y\|\}$. Obviously $S^{'}$ is closed, convex and nonempty ($Tx \in S^{'}$), then $z \in S$ and hence $Tz \in S^{'}$ by (2.2.1). Hence $S^{'}$ is invariant under T. Also

$$\delta(S^{'}) \leq \sup_{y \in S} \|y - Ty\| < \delta(S)$$

by hypothesis.

Hence $S^{'}$ is a proper subset of S, which contradicts the minimality of S.

Hence S has only one element which is a fixed point of T. The unicity of the fixed point follows from the fact if $x = Tx, y = Ty$ then

$$\|x - y\| = \|Tx - Ty\| \leq \frac{\|x - Tx\|}{2} + \frac{\|y - Ty\|}{2} = 0$$

i.e.; $x = y$. ■

Theorem 2.2.2 ***(Kannan)***

Let T be a continuous mapping of a bounded closed and convex subset K of a reflexive Banach space X into itself and let T have properties A and B over K. Then T

has a unique fixed point in K.

Proof.

Let S be the minimal element in X with respect to being closed, convex, bounded and invariant under T.

If S contains only one element, the theorem is obvious. If not, by property B, there exists $x \in S$, such that

$$\|x - Tx\| = r < \sup_{y \in S} \|y - Ty\| . \tag{2.2.2}$$

Let $P = \{x \in S : \|x - Tx\| \leqq r\}$. If $x \in P$, then since

$$\left\|Tx - T^2x\right\| \leq \frac{\|x - Tx\|}{2} + \frac{\|Tx - T^2x\|}{2}$$

we have $\|Tx - T^2x\| \leq r$ which implies $T(P) \subset P.LetP^{'}) = cl\ co(TP)$.

If $z \in P^{'}$, then any one of the following three cases may arise:

(1) $z \in TP$ and since $TP \subset P$, hence$Tz \in P^{'}$.

(2) $z = \sum_1^n \alpha_i T z_{i,} \alpha_i \geqq 0, \sum \alpha_i = 1$ and $z_i \in P$.

$$\begin{aligned}
\|z - Tz\| &= \left\|(\sum \alpha_i T z_i) - Tz\right\| \leq \sum \alpha_i \|Tz_i - Tz\| \\
&\leqq \sum \alpha_i \{\frac{\|z - Tz\|}{2} + \frac{\|z_i - Tz_i\|}{2}\} \\
&\leqq \sum \alpha_i \{\frac{\|z - Tz\|}{2} + \frac{r}{2}\}, \text{ since } \mathrm{z}_i \in \mathrm{P}, \\
&= \frac{\|z - Tz\|}{2} + \frac{r}{2}.
\end{aligned}$$

$\|z - Tz\| \leq r$ which implies $z \in P$ and hence$Tz \in Tz \in TP \subset P^{'}$

(3) z is a limit point of $P^{'}$, in which case by the continuity of T it follows that $z \in P$ and hence $Tz \in P^{'}$.

Thus $P^{'}$ is a closed, convex subset of S which is invariant under T and, for every element z of $P^{'}$, $\|z - Tz\| \leq r$, which implies by (2.2.2) that $P^{'}$ is a proper subset of S.

This contradicts the minimality of S. Hence S contain only one element. This element is the unique fixed point of T, unicity being true as seen as in theorem (2.2.1). ■

Theorem 2.2.3 ***(Banach)***

Let $(E, \|.\|)$ be a Banach space, and let $D \subseteq E$ be nonempty and closed. If the function $T : D \to D$ satisfies

$$\|T(x) - T(y)\| \leq q \cdot \|x - y\| \text{ for all } x, y \in D$$

with some $q < 1$, then within D there exists a unique fixed point x^* of T.

Proof.

First, we note that from (2.2.3) it follows that T is continuous. Then, we choose arbitrarily $x^{(0)} \in D$ and consider the sequence $x^{(n+1)} := T(x^{(n)})$, $n = 0, 1, ...$
This sequence is well-defined, since T is a self-mapping. From

$$\left\|x^{(n+1)} - x^{(n)}\right\| = \left\|T(x^{(n)}) - T(x^{(n-1)})\right\| \leq q \cdot \ \|x\| .$$

We get by induction

$$\left\|x^{(n+1)} - x^{(n)}\right\| \leq q^n \cdot \left\|x^{(1)} - x^{(0)}\right\| \tag{2.2.3}$$

Now, for any $p > 0$ we have

$$\begin{aligned}
\left\|x^{(n+p)} - x^{(n)}\right\| &\leq \left\|x^{(n+p)} - T(x^{(n+p)})\right\| + \left\|T(x^{n+p)}) - T(x^{(n)})\right\| + \left\|T(x^{(n)}) - x^{(n)}\right\| \\
&\leq \left\|x^{(n+p)} - T(x^{(n+p)})\right\| + q \cdot \left\|x^{(n+p)} - x^{(n)}\right\| + \left\|T(x^{(n)}) - x^{(n)}\right\| \\
&= \left\|x^{(n+p)} - x^{(n+p+1)}\right\| + q \cdot \left\|x^{(n+p)} - x^{(n)}\right\| + \left\|x^{(n+1)} - x^{(n)}\right\| .
\end{aligned}$$

So, by (2.2.4), it follows

$$\left\|x^{(n+p)} - x^{(n)}\right\| \leq \frac{1}{1-q}(q^{n+p} - q^n) \cdot \left\|x^{(1)} - x^{(0)}\right\| = \frac{q^p + 1}{1-q} \cdot \left\|x^{(1)} - x^{(0)}\right\| \cdot q^n \leq c \cdot q^n$$

with

$$c = \frac{2}{1-q} \cdot \left\|x^{(1)} - x^{(0)}\right\| .$$

Hence, $\{x^n\}_{n=0}^{\infty}$ is a Cauchy sequence. The latter has a limit, say x^*, since E is a Banach space, and $x^* \in D$, since D is closed. Due to the continuity of T we have

$$T(x^*) = T(\lim x^{(n)}) = \lim(T(x^{(n)}) = \lim x^{(n+1)} = x^*.$$

This means, that $x^* \in D$ is a fixed point of T.

To show uniqueness, we assume that T has two fixed points, say x^*, y^*. Then, with (2.2.3) it follows

$$\|x^* - y^*\| = \|T(x^*) - T(y^*)\| \leq q \cdot \|x^* - y^*\| .$$

However, this can only be true if $x^* = y^*$, since $q < 1$. To get the related error estimate we consider

$$\begin{aligned} \left\|x^{(n)} - x^*\right\| &\leq \left\|x^{(n)} - x^{(n+1)}\right\| + \left\|x^{(n+1)} - x^*\right\| \\ &= \left\|x^{(n+1)} - x^{(n)}\right\| + \left\|T(x^{(n)}) - T(x^*)\right\| \\ &\leq \left\|x^{(n+1)} - x^{(n)}\right\| + q \cdot \left\|x^{(n)} - x^*\right\| . \end{aligned}$$

hence,

$$\left\|x^{(n)} - x^*\right\| \leq \frac{1}{1-q} \cdot \left\|x^{(n+1)} - x^{(n)}\right\| .$$

By (2.2.3), we find

$$\left\|x^{(n)} - x^*\right\| \leq \frac{q^n}{1-q} \left\|x^{(1)} - x^{(0)}\right\| .$$

■

2.3 In the case of metric and generalized metric space

Theorem 2.3.1 ***(Banach)***

Let (X, d) be a complete metric space and let $F : X \to X$ be a contraction with Lipschitzian constant L. Then F has a unique fixed point $u \in X$. Furthermore, for any $x \in X$ we have

$$\lim_{n\to\infty} F^n(x) = u,$$

with

$$d(F^n(x), u) \leq \frac{L^n}{1-L} d(x, F(x)).$$

Proof.

We first show uniqueness. Suppose there exist $x, y \in X$ with $x = F(x)$ and $y = F(y)$. Then therefore $d(x, y) = 0$.

$$d(x, y) = d(F(x), F(y)) \leq Ld(x, y),$$

therefore $d(x, y) = 0$.

To show existence select $x \in X$. We first show that $\{F(x)\}$ is a Cauchy sequence. Notice for $n \in \{0, 1, ...\}$that

$$d(F^n(x), F^{n+1}(x)) \leq L \cdot d(F^{n-1}(x), F^n(x)) \leq ... \leq L^n \cdot d(x, F(x)).$$

Thus for $m < n$ where $n \in \{0, 1, ...\}$,

$$\begin{aligned} d(F^n(x), F^m(x)) &\leq d(F^n(x), F^{n+1}(x)) + d(F^{n+1}(x), F^{n+2}(x)) + ... + d(F^{m-1}(x), F^m(x)) \\ &\leq L^n \cdot d(x, F(x)) + ... + d(F^{m-1}(x), F^m(x)) \\ &\leq L^n \cdot d(x, F(x))[1 + L^2 + L^3 + L^4 + ...] \\ &\leq \frac{L^n}{1-L} \cdot d(x, F(x)). \end{aligned}$$

That is for $m > n, n \in \{0, 1, ...\}$,

$$d(F^n(x), F^m(x)) \leq \frac{L^n}{1-L} \cdot d(x, F(x)).$$

This shows that $\{F^n(x)\}$ is a Cauchy sequence and since X is complete there exists $u \in X$ with $\lim_{n\to\infty}(x) = u$.

Moreover the continuity of F yields

$$u = \lim_{n\to\infty} F^{n+1}(x) = \lim_{n\to\infty} F(F^n(x)) = F(u),$$

therefore u is a fixed point of F. ■

Remark 2.3.1 *The last theorem requires that $L < 1$. If $L = 1$ then F need not have a fixed point.*

Theorem 2.3.2 ***(Banach)***

Let (X, d) be a compact metric space with $F : X \to X$ satisfying

$$d(F(x), F(y)) < d(x, y)$$

for $x, y \in X$ and $x = y$.

Then F has a unique fixed point in X.

Proof.

To show existence, notice the map $x \to d(x, F(x))$ attains its minimum, say at $x_0 \in X$. We have $x = F(x)$ since otherwise

$$d(F(F(x_0)), F(x_0)) < d(F(x_0), x_0)$$

a contradiction.

To show uniqueness suppose that there exists $x, y \in X$ such that $x = F(x), y = F(y)$, $d(F(x), F(y)) = d(x, y)$,

$$d(x, y) = d(F(x), F(y)) < d(x, y)$$

therefore $d(x, y) = 0 \Longrightarrow x = y$. ■

Theorem 2.3.3 ***(Kannan)***

Let E be a metric space with ρ as metric. Let T be a map of E into itself, such that

- $\rho[T(p), T(q)] \leq \alpha\{\rho[p, T(p)] + \rho[q, T(q)]\},\ 0 < \alpha < \frac{1}{2}, p, q \in E.$
- T is continuous at a point $\xi \in E$.
- There exists a point $x \in E$ such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to ξ.

Then ξ is the unique fixed point of T.

Proof.

Continuity at ξ of T implies that $\{T^{n_{i+1}}(x)\}$ converges to $T(\xi)$.

Suppose $\xi \neq T(\xi)$. We consider two open discs $S_1 = S_1(\xi, \eta)$ and $S_2 = S_2(T(\xi), \eta)$ centered at ξ and $T(\xi)$ respectively and of radius $\eta > 0$ where $\eta < \frac{1}{3}\rho[\xi, T(\xi)]$.

Since$\{T^{n_i}(x)\}$ converges to ξ and $\{T^{n_{i+1}}(x)\}$ converges to $T(\xi)$, there exists a positive integer N_1 such that $i > N_1$ implies

$$T^{n_i}(x) \in S_1,\ T^{n_{i+1}}(x) \in S_2.$$

Hence

$$\rho[T^{n_i}(x), T^{n_{i+1}}(x)] > \eta, \quad (i > N_1).$$

On the other hand,

$$\rho[T^{n_{l+1}}(x), T^{n_{l+2}}(x)] \leqq \alpha\{\rho[T^{n_i}(x), T^{n_{i+1}}(x)] + \rho[T^{n_{i+1}}(x), T^{n_{i+2}}(x)]\}.$$

Hence

$$\rho[T^{n_{i+1}}(x), T^{n_{i+2}}(x)] \leqq \frac{\alpha}{1-\alpha}\rho[T^{n_i}(x), T^{n_{i+1}}(x)].$$

For $l > j > N_1$,we have

$$\begin{aligned}
\rho[T^{n_i}(x), T^{n_{i+1}}(x)] &\leqq \frac{\alpha}{1-\alpha}\rho[T^{n_{l-1}}(x), T^{n_l}(x)] \\
&\leqq (\frac{\alpha}{1-\alpha})^2\rho[T^{n_{l-2}}(x), T^{n_{l-1}}(x)] \\
&\leqq (\frac{\alpha}{1-\alpha})^{n_l - n_j}\rho[T^{n_i}(x), T^{n_{i+1}}(x)].
\end{aligned}$$

■

Theorem 2.3.4 ***(kannan)***

Let E be a metric space with ρ as metric and let T be a map of E into it self. Suppose that T is continuous at a point $x_0 \in E$. If there exists a point $x \in E$ such that the sequence of iterates $\{T^n(x)\}$ converges to x$_0$ then $T(x_0) = x_0$. If in addition,

$$\rho[T(x_0), T(\xi)] \leqq \alpha\rho(x_0, \xi),\ \xi \in E,\ 0 < \alpha < 1,$$

then x_0 is the unique fixed point of T.

Proof.

Let $x_n = T^n(x)$, then $x_n \to x_0$ as $n \to \infty$. Now

$$\begin{aligned} \rho[(x_0), x_0] &\leqq \rho[T(x_0), x_n] + \rho(x_n, x_0) \\ &= \rho[T(x_0), T(x_{n-1})] + \rho(x_n, x_0). \end{aligned}$$

The laft hand side is indipendent of n and the right hand side tends to zero as $n \to \infty$, so letting $n \to \infty$ we obtain $T(x_0) = x_0$.

If there exists $\delta \neq x_0$ such that $T(\delta) = \delta$ then

$$\rho(x_0, \delta) = \rho[T(x_0), T(\delta)] \leqq \alpha\rho(x_0, \delta),$$

gives $1 \leqq \alpha$, which is a contradiction. ■

Theorem 2.3.5 ***(Kannan)***

Let (X, d) be a complete metric space and $T, S : X \to X$ be a mapping such that T is continuous, one-to-one and subsequentially convergent. If $\lambda \in [0, \frac{1}{2})$ and

$$d(TSx, TSy) \leq \lambda[d(Tx, TSx) + d(Ty, TSy)] \quad (x, y \in X) \tag{2.3.1}$$

then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Proof.

Let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_{n+1} = Sx_n$ (equivalently, $x_n = S^n x_0$), $n = 1, 2, ...$Using (2.3.1), we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TSx_{n-1}, TSx_n) \\ &\leq \lambda[d(Tx_{n-1}, TSx_{n-1}) + d(Tx_n, TSx_n)], \end{aligned} \tag{2.3.2}$$

so,

$$d(Tx_n, Tx_{n+1}) \leq \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n). \tag{2.3.3}$$

Using induction and (2.3.3),

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n) \leq \left(\frac{\lambda}{1-\lambda}\right)^2 d(Tx_{n-2}, Tx_{n-1}) \\ &\leq ... \leq \left(\frac{\lambda}{1-\lambda}\right)^n d(Tx_0, Tx_1). \end{aligned} \tag{2.3.4}$$

By (2.3.4), for every $m, n \in N$ such that $m > n$ we have,

$$\begin{aligned} d(Tx_m, Tx_n) &\leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + ... + d(Tx_{n+1}, Tx_n) \\ &\leq \left[\left(\frac{\lambda}{1-\lambda}\right)^{m-1} + \left(\frac{\lambda}{1-\lambda}\right)^{m-2} + ... + \left(\frac{\lambda}{1-\lambda}\right)^{n}\right] d(Tx_0, Tx_1) \\ &= \left(\frac{\lambda}{1-\lambda}\right)^{n} \frac{1}{1-\left(\frac{\lambda}{1-\lambda}\right)} d(Tx_0, Tx_1). \end{aligned} \tag{2.3.5}$$

Letting $m, n \to \infty$ in (2.3.5), we have $\{Tx_n\}$ is a cauchy sequence, and since X is a complete metric space, there exists $v \in X$ such that

$$\lim_{n\to\infty} Tx_n = v. \tag{2.3.6}$$

Since T is a subsequentially convergent, $\{x_n\}$ has a convergent subsequence.

So there exists $u \in X$ and $\{x_{n(k)}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_{n(k)} = u$.

By (2.3.6) we conclude that $Tu = v$. So

$$\begin{aligned} d(TSu, Tu) &\leq d(TSu, TS^{n(k)}x_0) + d(TS^{n(k)}x_0, TS^{n(k)+1}x_0) + d(TS^{n(k)+1}x_0, Tu) \\ &\leq \lambda\left[d(Tu, TSu) + d(TS^{n(k)-1}x_0, TS^{n(k)}x_0)\right] \\ &\quad + \left(\frac{\lambda}{1-\lambda}\right)^{n(k)} d(TSx_0, Tx_0) + d(Tx_{n(k)+1}, Tu) \\ &\leq \lambda d(Tu, TSu) + \lambda\left(\frac{\lambda}{1-\lambda}\right)^{n(k)-1} d(Tx_0, Tx_1) + \\ &\quad \left(\frac{\lambda}{1-\lambda}\right)^{n(k)} d(Tx_1, Tx_0) + d(Tx_{n(k)+1}, Tu) \end{aligned}$$

hence,

$$\begin{aligned} d(TSu, Tu) &\leq \left(\frac{\lambda}{1-\lambda}\right)^{n(k)} d(Tx_0, Tx_1) + \frac{1}{1-\lambda}\left(\frac{\lambda}{1-\lambda}\right)^{n(k)} d(Tx_1, Tx_0) \\ &\quad + \frac{1}{1-\lambda} d(Tx_{n(k)+1}, Tu). \end{aligned} \tag{2.3.7}$$

Letting $k \to \infty$ in (2.3.7) we get $d(TSu, Tu) = 0$.

Since T is one-to-one $Su = u$. So S has a fixed point.

Uniqueness of the fixed point follows from (2.3.2).

Also if T is sequentially convergent, by replacing $\{n\}$ with $\{n(k)\}$ we conclude that $\lim_{n\to\infty} x_n = u$ and this shows that $\{x_n\}$ converges to the fixed point of S. ∎

Theorem 2.3.6 ***(Kannan)***

Let (X, d) be a complete generalized metric space, and the mapping $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)], \tag{2.3.8}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$.

Then T has a unique fixed point.

Proof.

Let x_0 be an arbitrary point in X. Let $x_1 = T(x_0)$, if $x_1 = x_0$ then $x_0 = T(x_0)$ this means x_0 is a fixed point of T and there is nothing to prove.

Assume that $x_1 \neq x_0$, let $x_2 = T(x_1)$. in this way we can define a sequence of points in X as follows:

$$x_{n+1} = Tx_n = T^{n+1}x_0, x_n \neq x_{n+1} \quad n = 0, 1, 2, ...$$

Using the inequality (2.3.8), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \lambda[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &\leq [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\leq \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n). \end{aligned}$$

We can also suppose that x_0 is not a periodic point, in fact if $x_n = x_0$, then,

$$\begin{aligned} d(x_0, Tx_0) &= d(x_n, Tx_n) = d(T^n x_0, T^{n+1}x_0) \leq \frac{\lambda}{1-\lambda} d(T^{n-1}x_0, T^n x_0) \\ &\leq \left[\frac{\lambda}{1-\lambda}\right]^2 d(T^{n-2}x_0, T^{n-1}x_0) \leq ... \leq \left[\frac{\lambda}{1-\lambda}\right]^n d(x_0, Tx_0). \end{aligned}$$

Put $h = \left[\frac{\lambda}{1-\lambda}\right]$, then $h < 1$ and

$$[1 - h^n]\, d(x_0, Tx_0) \leq 0.$$

It follows that x_0 is a fixed point of T. Thus in the sequel of proof we can suppose $T^n x_0$ for $n = 1, 2, 3, ...$Now ineguality (2.3.8) implies that

$$\begin{aligned} d(T^n x_0, T^{n+m}x_0) &\leq \lambda[d(T^{n-1}x_0, T^n x_0) + d(T^{n+m-1}x_0, T^{n+m}x_0)]. \\ &\leq \lambda[h^{n-1} d(x_0, Tx_0) + h^{n+m-1} d(x_0, Tx_0). \end{aligned}$$

Therefore, $d(x_n, x_{n+m}) \to 0$ as $n \to \infty$. It implies that $\{x_n\}$ is a cauchy sequence in X. Since X is complete, there exists a $u \in X$ such that $x_n \to u$. By rectangular

property we have

$$\begin{aligned}
d(Tu, u) &\leq d(Tu, T^n x_0) + d(T^n x_0, T^{n+1} x_0) + d(T^{n+1} x_0, u) \\
&\leq \lambda[d(u, Tu) + d(T^{n-1} x_0, T^n x_0)] + h^n d(x_0, Tx_0) + d(T^{n+1} x_0, u) \\
&\leq \lambda d(T^{n-1} x_0, T^n x_0) + \frac{h^n}{1-\lambda} d(x_0, Tx_0) + \frac{1}{1-\lambda} d(T^{n+1} x_0, u) \\
&\leq h^n d(x_0, Tx_0) + \frac{h^n}{1-\lambda} d(x_0, Tx_0) + \frac{1}{1-\lambda} d(T^{n+1} x_0, u).
\end{aligned}$$

Letting $n \to \infty$ and using the fact that, $d(\alpha_n, y) \to d(\alpha, y)$ and $d(x, \alpha_n) \to d(x, \alpha)$ whenever α_n is a sequence in X with $\alpha_n \to \alpha \in X$, we have $u = Tu$.

Now we show that T has a unique fixed point. For this, assume that there exists another point v in X such that $v = Tv$. Now,

$$\begin{aligned}
d(v, u) &= d(Tv, Tu) \\
&\leq \lambda d(v, Tv) + d(u, Tu) \\
&\leq \lambda d(v, v) + d(u, u) = 0.
\end{aligned}$$

Hence, $u = v$. ■

Chapter 3

Applications of Banach and Kannan's theorems

3.1 Introduction

In this chapter we discuss some applications of the theory of Banach and Kannan's to integral equations.The goal is to illustrate possibilities of applications of techniques of them.

3.2 Application in simple function

3.2.1 Applications of the principle of Banach

Let (M, d) be a metric space.

Let $T : M \to M$ be a continous function and contractante such that $T(x) = x^2 - 3$.

Let $x_0 = 2$, we have

We have

$$\begin{aligned}
x_0 &= 2, \\
x_1 &= \frac{1}{2}(x_0 + \frac{3}{x_0}) = \frac{7}{4} = 1.75, \\
x_2 &= \frac{1}{2}(x_1 + \frac{3}{x_1}) = 1.7321, \\
x_3 &= \frac{1}{2}(x_2 + \frac{3}{x_2}) = 1.7320, \\
x_4 &= \frac{1}{2}(x_3 + \frac{3}{x_3}) = 1.7320 = \sqrt{3}.
\end{aligned}$$

So $x = \sqrt{3}$ is the fixed point of T.

3.2.2 Applications of the principle of kannan

Consider $T : [0, 1] \to [0, 1]$ defined as

$$T(x) = \left\{ \begin{array}{cc} 1 - x & x \in \left[0, \frac{1}{3}\right) \\ \frac{x+1}{3} & x \in \left[1, \frac{1}{3}\right] \end{array} \right\}$$

For any $x, y \in [0, 1]$, $d(x, y) = |x - y|$.

If $x, y \in \left[0, \frac{1}{3}\right)$, then $d(Tx, Ty) = |Tx - Ty| = |x - y|$ and

$$d(x, Tx) = |x - Tx| = |2x - 1| \quad \text{for all } x.$$

$$\begin{aligned}
\frac{1}{2}(d(x, Tx) + d(y, Ty)) &= \frac{1}{2}(|2x - 1 + 2y - 1| \\
&= \left|x - \frac{1}{2}\right| + \left|y - \frac{1}{2}\right| \\
&\geq |x - y| = |(1 - y) - (1 - x)| \\
&= d(Tx, Ty).
\end{aligned}$$

This implies

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Tx) + d(y, Ty))$$

for all $x, y \in \left[0, \frac{1}{3}\right)$.

If $x, y \in \left[1, \frac{1}{3}\right]$,then $d(Tx, Ty) = |Tx - Ty| = \frac{1}{3}|x - y|$ and

$$d(x, Tx) = |x - Tx| = \frac{1}{3}|2x - 1|$$

for all x.

$$\begin{aligned}
\frac{1}{2}(d(x, Tx) + d(y, Ty)) &= \frac{1}{2}(\frac{1}{3}|2x - 1 + 2y - 1|) \\
&= \frac{1}{3}(\left|x - \frac{1}{2}\right| + \left|y - \frac{1}{2}\right|) \\
&\geq \frac{1}{3}|x - y| = \frac{1}{3}|(1 - y) - (1 - x)| = d(Tx, Ty).
\end{aligned}$$

This implies

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Tx) + d(y, Ty))$$

for all $x, y \in \left[1, \frac{1}{3}\right]$.

Thus T is a Kannan mapping have a unique fixed point.

3.3 Application in integral equation

3.3.1 Applications of the principle of Banach

Fredholm Integral Equations

Theorem 3.3.1 *If A is a bounded linear operator on a Banach space E, and φ is an arbitrary element of E, then the operator defined by*

$$Tf = \alpha Af + \varphi, \tag{3.3.1}$$

has a unique fixed point for any sufficiently small $|\alpha|$ .More precisely, if k is a positive constant such that

$$\|Af\| \leq k\|f\| \quad \textit{for all} \quad f \in E.$$

then $Tf = f$ has a unique solution whenever $|\alpha| k < 1$.

Proof.

Since A is bounded, there exists a constant k such that

$$\|Af_1 - Af_2\| \leq k\,\|f_1 - f_2\| \text{ for all } f_1, f_2 \in E.$$

Thus

$$\|Tf_1 - Tf_2\| = |\alpha|\,\|Af_1 - Af_2\| \leq |\alpha|\,|k|\,\|f_1 - f_2\|\,,$$

and hence T is a contraction whenever $|\alpha| < \frac{1}{k}$.

In such a case T has a unique fixed point by Theorem (2.2.3).

When the iterative process is applied in the case described in the previous theorem, we obtain the following sequence approximating the solution:

$$\begin{aligned}
f_0 &= \text{an arbitrary element of } E,\\
f_1 &= Tf_0 = \alpha Af_0 + \varphi,\\
f_2 &= T(\alpha Af_0 + \varphi) = \alpha^2 A^2 f_0 + \alpha A\varphi + \varphi,\\
&\vdots\\
f_n &= \alpha^n A^n f_0 + \alpha^{n-1}A^{n-1}\varphi + ... + \alpha^2 A^2 f_0 + \alpha A\varphi + \varphi,\\
&\vdots
\end{aligned}$$

Therefore, the solution f can be written as

$$f = \alpha A\varphi + \varphi + \alpha^2 A^2\varphi + \cdots + \alpha^n A^n\varphi + \cdots\,. \tag{3.3.2}$$

Note that the choice of f_0 is irrelevant for the solution. However, since some choices of f_0 give faster convergence of the series, in applications it may be important to make a good "first guess". Formally, the expansion (3.3.2) can be obtained directly from the equation

$$f - \alpha Af = \varphi$$

by expanding $(I - \alpha A)^{-1}$ into a geometric series

$$(I - \alpha A)^{-1} = I + \alpha A + \alpha^2 A^2 + \cdots\,.$$

■

Theorem 3.3.2 ***(Fredholm linear integral equation)***

The equation

$$f(x) = \alpha \int_a^b K(x,y) f(y) dy + \varphi(x)$$

has a unique solution $f \in L^2([a,b])$ provided the kernel K is continuous in $[a,b] \times [a,b], \varphi \in L^2([a,b])$, and $|\alpha|\, k < 1$, where

$$k = \sqrt{\int_a^b \int_a^b |K(x,y)|^2 \, dx dy}.$$

Proof.

Consider the operator

$$(Tf)(x) = \alpha \int_a^b K(x,y) f(y) dy + \varphi(x).$$

Since $\varphi \in L^2([a,b]), Tf \in L^2([a,b])$, if

$$\int_a^b K(x,y) f(y) dy \in L^2([a,b]). \tag{3.3.3}$$

By Schwarz's inequality, we find

$$\begin{aligned} \left| \int_a^b K(x,y) f(y) dy \right|^2 &\leq \int_a^b |K(x,y) f(y)| \, dy \\ &\leq \left(\int_a^b |K(x,y)|^2 \, dy \right)^{\frac{1}{2}} \left(\int_a^b |f(y)|^2 \, dy \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\left| \int_a^b K(x,y) f(y) dy \right|^2 \leq \left(\int_a^b |K(x,y)|^2 \, dy \right)^{\frac{1}{2}} \left(\int_a^b |f(y)|^2 \, dy \right)^{\frac{1}{2}}$$

and

$$\begin{aligned} \int_a^b \left| \int_a^b K(x,y) f(y) dy \right|^2 dx &\leq \int_a^b \left(\int_a^b |K(x,y)|^2 \, dy \int_a^b |f(y)|^2 \, dy \right) dx \\ &\leq \int_a^b \int_a^b |K(x,y)|^2 \, dy dx \int_a^b |f(y)|^2 \, dy. \end{aligned}$$

Since

$$\int_a^b \int_a^b |K(x,y)|^2 \, dydx < \infty \quad \text{and} \quad \int_a^b |f(y)|^2 \, dy < \infty,$$

(3.3.4) is satisfied and thus, T maps $L^2([a,b])$ into it self.

Note that the above shows also that the operator defined by

$$(Af)(x) = \int_a^b K(x,y)f(y)dy,$$

is bounded. Therefore, by the Theorem (3.3.1), the equation $Tf = f$ has a unique solution whenever $|\alpha|k < 1$. ■

Theorem 3.3.3 ***(Fredholm non-linear integral equation)***

Suppose

- $\left\| \int_a^b k(x,y,f(y))dy \right\| \leq M \|f\| \;\; forall \;\; f \in L^2([a,b]),$
- $|k(x,y,z_1) - k(x,y,z_2)| \leq N(x,y) |z_1 - z_2|$ for all $x, y, z_1, z_2 \in [a,b]$
- $\int_a^b \int_a^b |N(x,y)|^2 \, dxdy < k^2 < \infty.$

Then the nonlinear Fredholm equation

$$f(x) = \alpha \int_a^b K(x,y)f(y))dy + \varphi(x) \tag{3.3.4}$$

has a unique solution $f \in L^2([a,b])$ for every $\varphi \in L^2([a,b])$ and every α such that $|\alpha| k < 1$.

Proof.

Consider the operator

$$Tf = \alpha Af + \varphi$$

where

$$(Af)(x) = \int_a^b k(x,y,f(y))dy.$$

Then

$$\begin{aligned}
\|Tf_1 - Tf_2\| &= |\alpha| \left\| \int_a^b (k(x,y,f_1(y) - k(x,y,f_2(y))dy \right\| \\
&\leq |\alpha| \left(\int_a^b \left(\int_a^b |(k(x,y,f_1(y) - k(x,y,f_2(y))| \, dy \right)^2 dx \right)^{\frac{1}{2}} \\
&\leq |\alpha| \left(\int_a^b \left(\int_a^b N(x,y) \, |f_1(y) - f_2(y)| \, dy \right)^2 dx \right) \\
&\leq |\alpha| \, k \, \|f_1 - f_2\| \, .
\end{aligned}$$

Clearly, if $|\alpha| \, k < 1$, then T is a contraction operator and thus it has a unique fixed point. That fixed point is a solution of Equation (3.3.5). ■

Example 3.3.1

Solve the Fredholm integral equation by using the successive approximations method

$$\varphi(x) = x + e^x - \int_0^1 xt\varphi(t)dt. \tag{3.3.5}$$

For the zeroth approximation $\varphi_0(x)$, we can select

$$\varphi_0(x) = 0. \tag{3.3.6}$$

The method of successive approximations admits the use of the iteration formula

$$\varphi_{n+1}(x) = x + e^x - \int_0^1 xt\varphi_n(t)dt, \; n \geq 0. \tag{3.3.7}$$

Substituting (3.3.7) into (3.3.8) we obtain

$$\begin{aligned}
\varphi_1(x) &= x + e^x - \int_0^1 xt\varphi_0(t)dt = x + e^x, \\
\varphi_2(x) &= x + e^x - \int_0^1 xt\varphi_1(t)dt = e^x - \frac{1}{3}x, \\
\varphi_3(x) &= x + e^x - \int_0^1 xt\varphi_2(t)dt = e^x + \frac{1}{9}x, \\
&\vdots \\
\varphi_{n+1}(x) &= x + e^x - \int_0^1 xt\varphi_n(t)dt = e^x + \frac{(-1)^n}{3^n}x.
\end{aligned}$$

Consequently, the solution $\varphi(x)$ of (3.3.6) is given by

$$\varphi(x) = \lim_{n\to\infty} \varphi_{n+1}(x) = e^x.$$

Volterra Integral Equations

Suppose $\varphi \in L^2([a,b])$ and the kernel K satisfies the condition

$$\int_a^b\int_a^b |K(x,y)|^2\, dxdy < \infty. \tag{3.3.8}$$

Then the equation

$$f(x) = \alpha\int_a^x K(x,y)f(y)dy + \varphi(x) \tag{3.3.9}$$

has a unique solution in $L^2([a,b])$ for arbitrary $\alpha \in \mathbb{C}$. The solution can be written in the form

$$f(x) = \varphi(x) + \sum_{n=1}^{\infty} \alpha^n \int_a^x K_n(x,t)\varphi(t)dt, \tag{3.3.10}$$

where the kernels $K_n(x,t)$ satisfy the recurrence relation

$$\begin{aligned}
K_1(x,t) &= K(x,t) \qquad (3.3.11)\\
K_n(x,t) &= \int_a^x K(x,\xi)K_{n-1}(x,\xi)d\xi, \quad \text{for} \quad n \geq 2.
\end{aligned}$$

Proof. We set

$$A(x) = \int_a^x |K(x,y)|^2 \, dy \quad \text{and} \quad B(x) = \int_y^b |K(x,y)|^2 \, dx$$

By(3.3.9), A and B are integrable functions, so that there exists a constant M such that

$$\int_a^b A(x)dx \leq M \quad \text{and} \quad \int_a^b B(x)dy \leq M.$$

We also introduce the function λ on $[a,b]$ defined by

$$\lambda(x) = \int_a^x A(t)dt.$$

Clearly $0 \leq \lambda(x) \leq M$ for all $x \in [a,b]$.Consider the operator

$$(Tf)(x) = \alpha \int_a^x K(x,y)f(y)dy + \varphi(x).$$

We shall show that T^n is a contraction for some $n \in \mathbb{N}$, and then use Theorem(2.2.3) to conclude that T has a fixed point. That fixed point must be a unique solution of (3.3.10)If we write

$$Tf = \alpha Wf + \varphi$$

where

$$Wf(x) = \int_a^x K(x,y)f(y)dy,$$

then

$$T^n\varphi = \varphi + \alpha W\varphi + \alpha^2 W^2\varphi + \ldots + \alpha^n W^n \varphi.$$

The operators W^m can be written in the form

$$(W^m g)(x) = \int_a^x K_m(x,y)g(y)dy,$$

where the kernels K_n are defined by (3.3.12). Indeed, for $m = 2$ we have

$$(W^2 g)(x) = \int_a^x K(x,z) \int_a^x K(z,y)g(y)dydz.$$

This integral can be considered as a double integral over the triangular region $\{(y,z) : a \leq y \leq z \text{ and } a \leq z \leq x\}$ (see Figure 1). After interchangingthe order of integration, we obtain

$$(W^2g)(x) = \int_a^x \int_y^x K(x,z)K(z,y)dzg(y)dy.$$

If we denote

$$K_2(x,y) = \int_y^x K(x,z)K(z,y)dz,$$

then by a similar argument we get

$$(W^3g)(x) = \int_a^x \int_y^x K(x,z)K_2(z,y)dzg(y)dy,$$

and so on, as stated earlier.à

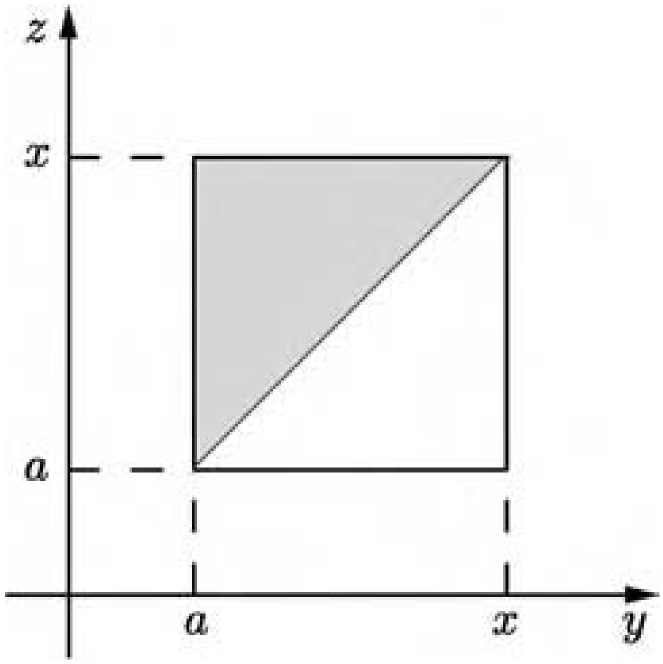

Figure 1:The triangular region
$\{(y,z) : \alpha \leq y \leq z$ and $\alpha \leq z \leq x\}$

To estimate$\|W^m\|$, we examine K_m. for $m=2$, application of Schwarz's inequality gives

$$\begin{aligned}|K_2(x,y)|^2 &= \left|\int_y^x K(x,z)K(z,y)dz\right|^2\\ &\le \int_y^x |K(x,z)|^2\,dz\int_y^x |K(z,y)|^2\,dz \le A(x)B(y).\end{aligned}$$

Similarly,

$$\begin{aligned}|K_3(x,y)|^2 &\le \int_y^x |K(x,z)|^2\,dz\int_y^x |K(z,y)|^2\,dz\\ &\le A(x)B(y)\int_a^x A(z)dz = A(x)B(y)(\lambda(x)-\lambda(y)).\end{aligned}$$

By induction, we can show that

$$|K_m(x,y)|^2 \le A(x)B(y)\frac{(\lambda(x)-\lambda(y))^{m-2}}{(m-2)!} \quad \text{for } m\ge 2.$$

Therefore,

$$\begin{aligned}|T^m f_1(x)-T^m f_2(x)|^2 &= |\alpha|^{2m}\left|\int_a^x K_m(x,y)(f_1(y)-f_2(y))dy\right|^2\\ &\le |\alpha|^{2m}\int_a^x \frac{A(x)B(y)[\lambda(x)-\lambda(y)]^{m-2}}{(m-2)!}dy\int_a^x |f_1(y)-f_2(y)|^2\,dy\\ &\le \frac{|\alpha|^{2m}A(x)(\lambda(x))^{m-2}}{(m-2)!}\int_a^x B(y)dy\,\|f_1-f_2\|^2\\ &\le \frac{|\alpha|^{2m}A(x)(\lambda(x))^{m-2}M}{(m-2)!}\|f_1-f_2\|^2.\end{aligned}$$

Integrating with respect to x in $[a,b]$, we obtain

$$\|T^m f_1(x)-T^m f_2(x)\|^2 \le \frac{|\alpha|^{2m}M^m}{(m-1)!}\|f_1-f_2\|^2 \quad \text{for } m\ge 2.$$

Therefore, since there exists $n\in\mathbb{N}$ such that

$$\frac{|\alpha|^{2n}M^n}{(n-1)!}<1,$$

T^n is a contraction. In view of Theorem (2.2.3) Equation (3.3.10) has a unique solution that can be written in the form

$$\lim_{n\to\infty} T^n f = \varphi + \alpha W\varphi + \alpha^2 W^2\varphi + \alpha^3 W^3\varphi + ...,$$

or, equivalently,

$$f(x) = \varphi(x) + \sum_{n=1}^{\infty} \alpha^n \int_a^x K_n(x,t)\varphi(t)dt.$$

■

Theorem 3.3.4 ***(Homogeneous Volterra equation)***

The homogeneous Volterra equation

$$f(x) = \alpha \int_a^x K(x,t) f(t)dt, \qquad x \in [0,1] \tag{3.3.12}$$

has only the trivial solution $f = 0$.

Proof. From (3.3.12), we have

$$|f(x)| \leq \alpha \int_0^x |K(x,t)|\,|f(t)|\,dt \leq \alpha M p, \tag{3.3.13}$$

where

$$p = \int_0^1 |f(t)|\,dt$$

and M is a constant such that $|K(x,t)| \leq M$ for all $x, t \in [0,1]$. Hence , by using (3.3.13) in (3.3.12), we obtain

$$|f(x)| \leq |\alpha| \int_0^x |K(x,t)|\,|\alpha|\,Mpdt \leq |\alpha|^2 M^2 px.$$

By continuing the process, we get

$$|f(x)| \leq |\alpha|^n M^n p \frac{x^{n-1}}{(n-1)!} \leq \frac{|\alpha|^n M^n p}{(n-1)!} \to 0 \text{ as } n \to \infty.$$

This shows $f(x) = 0$ for all $x \in [0,1]$.

■

Example 3.3.2

Solve the Volterra integral equation by using the successive approximations method

$$\varphi(x) = 1 - \int_0^x (x-t)\varphi(t)dt.$$

For the zeroth approximation $f_0(x)$, we can select

$$\varphi_0(x) = 1. \tag{3.3.14}$$

The method of successive approximations admits the use of the iteration formula

$$\varphi_{n+1}(x) = 1 - \int_0^x (x-t)\varphi_n(t)dt, n \geq 0. \tag{3.3.15}$$

Substituting (3.3.14) into (3.3.15) we obtain

Consequently, we obtain

$$\varphi_{n+1}(x) = \sum_{k=1}^{n} (-1)^k \frac{x^{2k}}{(2k)!}.$$

The solution $\varphi(x)$ is

$$\varphi(x) = \lim_{k\to\infty} \varphi_{n+1}(x) = \cos(x).$$

Conlusion

In this memory we treated the difference or equivalence between the two Banach and Kannan's principles on metrics spaces and functional spaces.

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ملخص

تعتبر نظريات النقطة الصامدة أدوات مهمة للغاية في التحليل الدالي في هذه المذكرة قمنا بدراسة الفرق بين مبدأ كل من بناخ و كنان على مفهوم النقطة الصامدة في فضائين مختلفين فضاء متري و فضاء دالي.

الكلمات المفتاحية: فضاء بناخ، فضاء متري، النقطة الصامدة،

Résumé

Les théorèmes du point fixe sont des outils très important dans l'analyse fonctionnelle, dans ce mémoire on étude la différence entre le principe de Banach et kannan dans l'espace métrique et l'espace fonctionnel.

Mots clés: Point fixe, espace métrique, espace de Banach, contractive ...

Abstract

Fixed-point theorems are very important tools in functional analysis, in this memory we study the different between the Banach and Kannan's principle in two cases metric spaces and functional spaces.

Key words: Fixed point, metric space, Banach space, contraction…